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A microscopic spiking neuronal network for the age-structured model

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Abstract We introduce a microscopic spiking network consistent with the age-structured/renewal equation proposed by Pakdaman, Perthame and Salort. It is a jump process interacting through a global activity variable with random delays. We show the well-posedness of the particle system and the mean-field equation. Moreover, by studying the tightness of the empirical measure, we prove the propagation of chaos property. Eventually, we quantify the rate of convergence under the assumption of compactly supported initial data.

Keywords Chaos propagation · mean-field limits · neuronal networks · age structured model · Poisson coupling.

Mathematics Subject Classification (2000) 82C22 · 82C44 · 37N25 · 60K35

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1 Introduction and main results

In a series of outstanding papers, Pakdaman, Perthame and Salort (PPS) [11, 12, 13] introduced a versatile model for the large-scale dynamics of neuronal networks. These equations describe the probability distribution of the time elapsed since the last spike fired as an age-structured nonlinear PDE. Inspired by the dynamics of these macroscopic equations, we propose here a microscopic model describing the dynamics of a finite number of neurons, and that provides a realistic neural network model consistent with the PPS model, in the sense that in the thermodynamic limit, propagation of chaos and convergence to the that equation is proved. Let $f = f(t, x) \geq 0$ be the density of neurons in the *state* $x \in \mathbb{R}_+$ at time $t \geq 0$. The dynamics of the age-structured PPS model are given by the following nonlinear integral and partial differential equation

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial x} + a(x, M(t)) f(t, x) \; = \; 0, \\ f(t, x = 0) = N(t), \end{cases} \tag{1}$$

where $N(t)$ and the network activity function $M(t)$ are defined by the relation

$$M(t) := \int_0^t b(y)\, N(t-y)\, dy, \quad N(t) := \int_0^\infty K(x, M(t))\, f(t, x)\, dx, \tag{2}$$

i.e., $M(t)$ is the convolution between the instantaneous total activity of the network $N(t)$ and a delay kernel $b(dy)$.

Using similar ideas, let us describe the neuronal network we propose. For any $N \geq 0$ we consider a set $\{1, \dots, N\}$ of interacting neurons. The state of each neuron i is described by a $\mathbb{R}_+$-valued variable $X_t^{i,N}$ corresponding to the time elapsed since last discharge. This approach is quite different from classical literature, where the key variable is the voltage, and constitutes an important originality of the PPS model. Neurons interact through the emission and reception of action potentials (or spikes), which are fast stereotyped trans-membrane current. The spiking rate essentially depends on the global activity M of the network. Specifically, a neuron with age x (duration since it fired its last spike) fires an action potential with an instantaneous intensity $a(x, M)$. Subsequently to the spike emission, two things happen: the age of the spiking neuron is reset to 0, and the global variable M increases its value by an extra value of J/N. The coefficient J represents a mean strength of connectivity of the network. When no spikes occur, the global variable M is supposed to decay exponentially to 0 at a constant rate α.

For each $N \in \mathbb{N}$, let us consider a family $(\mathcal{N}_t^1, \ldots, \mathcal{N}_t^N)_{t\geq 0}$ of i.i.d. standard Poisson processes and a family $(\tau_1, \ldots, \tau_N)$ of i.i.d. real valued random variables with probability law b. These coefficients represent delays in the transmission of information from the cell to whole network. Furthermore, we assume that the family of delays is independent of the Poisson processes.

Throughout the paper we assume chaotic initial conditions, in the sense that the initial states of the neurons are independent and identically distributed random variables. Therefore, for g_0 and m_0 two independent probability measures on $\mathbb{R}_+$, $(g_0 \otimes m_0)$-*chaotic initial states* consists in setting i.i.d. initial conditions for all neurons with common law equal to g_0, and setting independently, for the global activity variables, another i.i.d. initial values with common law equal to m_0.

Our aim is to understand the convergence of the $\mathbb{R}_+^2$-valued Markov processes

$$(X_t^N, M_t^N)_{t\geq 0} = ((X^{1,N}, M_t^{1,N}) \ldots, (X_t^{N,N}, M_t^{N,N}))_{t\geq 0},$$

solving, for each $i = 1, \ldots, N$ and any $t \geq 0$:

$$X_t^{i,N} = X_0^{i,N} + t - \int_0^t X_{s-}^{i,N} \int_0^\infty \mathbf{1}_{\{u \leq a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \mathcal{N}^i(du, ds), \qquad (3)$$

with the adapted set of coupling variables given by

$$\begin{aligned} M_t^{i,N} = M_0^{i,N} - \alpha \Big[\int_0^t M_s^{i,N}\, ds \\ - \frac{J}{N} \sum_{j \neq i} \int_0^{t-\tau_j} \int_0^\infty \mathbf{1}_{\{u \leq a(X_{s-}^{j,N}, M_{s-}^{j,N})\}} \mathcal{N}^j(du, ds) \Big]. \end{aligned} \qquad (4)$$

The presence of τ_j in the total activity variables is a consistency restriction on the spiking times: *when a neuron j sends a signal at a time $t \geq 0$, it is taken in consideration by the i-th global variable only with after a delay τ_j.*

Finally, we make some physically reasonable assumptions on the intensity spike function of the system:

$$\begin{cases} a(\cdot,\cdot) \text{ is a continuous non decreasing function in both variables,} \\ a(0,\cdot) \;=\; 0, \end{cases} \qquad (5)$$

representing that neurons have a higher probability of spike if the have been in repose for a long time or if the activity of the networks is high. We also impose a consistency restriction

$$(\exists\, \delta_0 > 0)(\forall\, \delta \in (0, \delta_0))(\exists\, x_\delta^* > 0) \text{ such that } a(x, m) \leq \delta, \quad \forall\, m \in \mathbb{R}_+ \qquad (6)$$

representing that, independently of the level of the network activity, a neuron cannot spike two times in an arbitrary small period of time.

For the previous setting, we have directly the

Proposition 1 *Under hypotheses (5) and (6), let $N \geq 1$ be fixed and assume that a.s.,*

$$\max_{1\leq i\leq N} |(X_0^{i,N}, M_0^{i,N})| < \infty,$$

then there exists a unique càdlàg adapted strong $\mathbb{R}_+^2$-valued solution $(X_t^N, M_t^N)_{t\geq 0}$ to (3)-(4).

Under suitable conditions (to be preciser later on) we show that the solution $(X_t^N)_{t\geq 0}$ behave, for large values of N, as N independent copies of the solution to a *nonlinear* SDE that we introduce now. Let (X_0, M_0) be a $(g_0 \otimes m_0)$-distributed random variable and $\mathcal{N}_t$ a standard Poisson process independent of X_0 and M_0. We analyse the existence and consistency of a $\mathbb{R}_+^2$-valued *càdlàg* adapted process $(X_t, M_t)_{t\geq 0}$ solving for any $t \geq 0$

$$X_t = X_0 + t - \int_0^t X_{s-} \int_0^\infty \mathbf{1}_{\{u\leq a(X_{s-}, M_{s-})\}} \mathcal{N}(du, ds), \tag{7}$$

and

$$M_t = M_0 - \alpha\Big[\int_0^t M_s\, ds - J\int_0^t \int_0^s \mathbb{E}[a(X_{s-w}, M_{s-w})]\, b(dw) ds\Big]. \tag{8}$$

Remark 2 *Let us assume for a moment an hypothesis of instantaneous membrane decay, i.e. $\alpha \to \infty$. In that case equation (8) writes*

$$M_t \;=\; J\int_0^t \mathbb{E}\big[a(X_{t-w}, M_{t-w})\big]\, b(dw),$$

in particular, M_t is a deterministic function of t. Then, if the probability density of X_t is given by $f_t(dx)$, the previous relation is reduced to

$$M(t) \;=\; J\int_0^t \int_0^\infty a(x, M(t-w))\, f_{t-w}(dx)\, b(dw).$$

Coming back to equation (7), we have that $f_t(x)$ solves in the weak sense equation (1)

The nonlinear SDE is clearly well-posed if we, for instance, make a Lipschitz continuity assumption on the intensity function. In order to avoid this simplification, here we try to use the approaches of Fournier-Löcherbach [5] and/or Robert-Touboul [15]. The second natural result of the manuscript is

Theorem 3 *Let us assume that hypotheses (5)-(6) hold, then there exists a weak solution $(X_t, M_t)_{t\geq 0}$ to (7)-(8) such that*

$$\int_0^t \int_0^s \mathbb{E}\big[a(X_{s-w}, M_{s-w})\big]\, b(dw)\, ds < \infty, \qquad \forall\, t \geq 0. \tag{9}$$

Moreover, if the law of (X_0, M_0) is compactly supported, then there exists a unique strong solution $(X_t, M_t)_{t\geq 0}$ to (7)-(8) in the class of functions such that there are deterministic locally bounded functions $A, B : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$\sup_{t\in[0,T]} \|X_t/A(t)\|_\infty < \infty, \qquad \sup_{t\in[0,T]} \|M_t/B(t)\|_\infty < \infty, \qquad \forall\, T \geq 0. \tag{10}$$

Under some extra assumptions on the growing of the intensity function, the existence and uniqueness result still hold true if the initial datum has a fast decay at infinite. More precisely, let us consider that

$$(\exists\, \xi > 2) \,:\, a(x,m) \leq C_\xi (1 + x^{\xi-2} + m^{\xi-2}), \tag{11}$$

and that the intensity function is such that there is a positive constant C_0 such that

$$|a(x,m) - a(x',m')| \,\leq\, C_0 a(x,m) \wedge a(x',m') \big[|x - x'| + |m - m'|\big], \tag{12}$$

for all $x, x', m, m' \in \mathbb{R}_+$.

Theorem 4 *Let us fix $\omega > 0$. There exists a unique strong solution $(X_t, M_t)_{t\geq 0}$ to (7)-(8) in the class of functions of locally bounded exponential moments:*

$$\sup_{t\in[0,T]} \mathbb{E}\big[e^{\omega(|X_t|^\xi + |M_t|^\xi))}\big] < \infty, \qquad \forall\, T > 0. \tag{13}$$

Finally we analyse the chaoticity of the system. To do so, a few more notations must be introduced. We denote by $\mathbb{D}(\mathbb{R}_+^2)$ the set of càdlàg functions on $\mathbb{R}_+^2$ endowed with the topology of the convergence on compact time intervals. By definition, each pair $(X_t^{i,N}, M_t^{i,N})_{t\geq 0}$ belongs to $\mathbb{D}(\mathbb{R}_+^2)$, and then the sequence of empirical measures

$$\mu_N = N^{-1} \sum_{i=1}^{N} \delta_{\{(X_t^{i,N}, M_t^{i,N})_{t\geq 0}\}},$$

is well defined and belongs to $\mathbb{P}(\mathbb{D}(\mathbb{R}_+^2))$. The third and last main result of the manuscript is

Theorem 5 *Let us assume that hypotheses (5)-(6) hold, and that the law of (X_0, M_0) is compactly supported, then the sequence of empirical processes $\mu_N(t)$ converges in distribution to the law of the process $(X_t, M_t)_{t\geq 0}$ with $(g_0 \otimes m_0)$-chaotic initial states solution to (7)-(8).*

If the initial datum has a fast decay (in the sense described in Theorem 4), and if moreover continuity assumption (12) holds, then the convergence of $\mu_N(t)$ remains true.

Mathematical overview. As we already said, the aim of the present work is to give a new microscopic point of view of the age structured equation considered in Pakdaman-Perthame-Salort [11,12,13]. Therein, the model is proposed as a reinterpretation of the well known renewal equation and the microscopic derivation is omitted. In Tanabe-Pakdaman [19] and Vibert-Champagnat-Pakdaman-Pham [22] authors propose a particle system but the question of convergence and chaos propagation is not addressed either. Nevertheless, the questions of existence of stationary solutions for the PDE and the numerical/simulation aspects of both: *the particle system and the limit equation*, are deeply studied and several very interesting results, regarding the existence of oscillatory solutions, are given. Moreover, the effects of the finite size of the populations are contrasted with the solutions of the limit equation.

From the mathematical point of view, we use the ideas of propagation of chaos which is a very well known and popular topic since the seminal works of Kac [8], McKean [9,10], and Sznitzman [17]. The idea of convergence and chaos propagation is classical: *when the number of particles is going to infinity, each one of them behaves as independent copies of the solution of a mean field equation*. The nonlinearity is characterized by the presence of the law itself in the dynamics on the process. In PDE terminology the problem is a integro-differential nonlinear equation.

The specific mathematical tools used in the present work can be easily traced down to two recent manuscripts addressing the question of chaoticity of a unidimensional model: Founier-Löcherbach [5] and Robert-Touboul [15]. The first paper solves the problem under the merely assumption of integrability on the initial condition, which is a remarkable weak hypothesis. Nevertheless, the path-wise uniqueness proof (which is at the end the key point of the method) uses a particular distance that is closely related to the equation itself. A different approach, based on the discretization of the limit equation, is presented in [3]. There, the convergence is proved imposing compactness on the support of the initial conditions. In [15], the authors also assume boundedness of the initial conditions, and provide a qualitative characterization of the qualitative properties of stationary solutions and their stability in the finite-size and mean-field systems, and a detailed discussion of bifurcations, stability and multiple stationary solutions is given.

To prove the chaos propagation property there are two classical approaches: whether we use the *coupling* method or an abstract *compactness* argument. The coupling is very intuitive and apply in a very wide range of applications. Nevertheless, usually it is assumed that functions involved are Lipschitz continuous (for a very nice review see [21]). Moreover, the method provides the rate of convergence by explicitly estimate the difference between the empirical measures. In Bolley-Cañizo-Carrillo [2], it is proved that the method still apply in the case of locally Lispchitz continuity, but imposing some exponential moment conditions. The second method is much more general and uses a more abstract framework. It was introduced by Sznitman [16], and it is useful to prove also the existence of solutions to the SDE, but do not provide the rate of convergence.

The present model has two main novelties. The first one is that the system is not one-dimensional but add an extra equation for the coupling variable. This issue implies in particular that attempts to use the distance of Founier-Löcherbach [5] fail unless a more suitable distance is found. Moreover, the two-dimensional nature of the empirical measures makes that the rate of convergence attained for a L^1 Wasserstein distance is lower than $N^{1/2}$. A perspective of the work is to use a combined PDE/SDE approach to find a sharper entropy function that allows us to have the necessary uniqueness result (see e.g. Godinho [6]). The second novelty of the present work is the presence of delays, that is central in the original PPS model. This issue is solved by using the independence of the random environment (see e.g. Touboul [20,14]), in particular, the extra terms are treated as locally square integrable processes.

Plan of the paper. The paper is organized as follows: Section 2 deals with the well-posedness of the particle system by finding some nice *a priori* bounds of the solutions. Section 3 is related to the path-wise uniqueness question of the mean field system. Well-posedness of the mean field system is studied in Section 4 and Section 5, in particular, we use a compactness argument to find that the sequence of empirical measures converges to a weak solution of the SDE. Finally in Section 6, we use the *coupling* method to study the rate of convergence. Appendix A completes the present work with some general well known results in stochastic calculus theory that are useful to our developments.

2 The particle system

Throughout the present section we fix the number of neurons $N \geq 1$. For $\mu \in \mathbb{P}(\mathbb{D}(\mathbb{R}_+^2))$ and $\varphi \in C(\mathbb{R}_+^2)$, let us denote the duality product,

$$\langle \mu(t), \varphi \rangle := \int_{\mathbb{D}(\mathbb{R}_+^2)} \varphi(\gamma_t, \beta_t)\, \mu(d\gamma, d\beta).$$

Lemma 6 *Any solution* $((X_t^{1,N}, M_t^{1,N}), \ldots, (X_t^{N,N}, M_t^{N,N}))_{t\geq 0}$ *to* (3)-(4) *satisfies a.s.*

$$\max_{1,\ldots,N} X_t^{i,N} \leq \max_{1,\ldots,N} X_0^{i,N} + t, \qquad \forall\, t \geq 0. \tag{14}$$

Moreover, there is a positive constant C_1*, depending only on the parameters of the system, such that for any* $t \geq 0$

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^t (1 + X_{s-}^{i,N}) \int_0^\infty \mathbf{1}_{\{u \leq a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \mathcal{N}^i(du, ds) \leq C_1(t + \bar{X}_0^N) + Z_t^N, \tag{15}$$

where

$$Z_t^N := \frac{1}{N}\sum_{j=1}^{N}\int_0^t \int_0^\infty \mathbf{1}_{\{u\leq \delta_0\}} \mathcal{N}^j(du, ds).$$

As a consequence, there is another positive constant C_1'*, such that a.s.*

$$\max_{1,\ldots,N} M_t^{i,N} \leq \max_{1,\ldots,N} M_0^{i,N} + C_1'(t + \bar{X}_0^N + Z_t^N), \quad \forall t \geq 0, \tag{16}$$

Proof Inequality (14) is direct from the definition of $X_t^{i,N}$ and the positivity of initial conditions. Moreover, denoting by $\bar{X}_t^N$ the (empirical) mean of $(X_t^{i,N})$, it follows that

$$\bar{X}_t^N = \bar{X}_0^N(0) + t - \frac{1}{N}\sum_{i=1}^{N}\int_0^t X_{s-}^{i,N} \int_0^\infty \mathbf{1}_{\{u\leq a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \mathcal{N}^i(du, ds),$$

and using that $\bar{X}_t^N$ is nonnegative,

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^t X_{s-}^{i,N}\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{i,N},M_{s-}^{i,N})\}}\mathcal{N}^i(du,ds)\;\le\;\bar{X}_0^N(0)+t.$$

Next, we fix some $\delta\in(0,\delta_0)$, and use the consistency condition (6) to get that

$$(\exists\, x_\delta^*)\;(\forall\, m\in\mathbb{R}_+, x\le x_\delta^*),\quad a(x,m)\le\delta_0, \tag{17}$$

in particular, using that $x\ge x_\delta^*\,(1-\mathbf{1}_{x<x_\delta^*})$, we have

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^t\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{i,N},M_{s-}^{i,N})\}}\mathcal{N}^i(du,ds)$$
$$\le\;\frac{1}{x_\delta^*}(\bar{X}_0^N+t)+\frac{1}{N}\sum_{i=1}^{N}\int_0^t\int_0^\infty \mathbf{1}_{\{u\le\delta_0\}}\mathcal{N}^i(du,ds),$$

and inequality (15) follows.

To control $(M_t^{i,N})_{t\ge0}$, we notice that equation (4) implies that

$$M_t^{i,N}\;\le\;M_0^{i,N}+\frac{\alpha\,J}{N}\sum_{j=1}^{N}\int_0^{t-\tau_j}\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{j,N},M_{s-}^{j,N})\}}\,\mathcal{N}^j(du,ds),$$

but a is nonnegative, therefore

$$M_t^{i,N}\;\le\;M_0^{i,N}+\frac{\alpha\,J}{N}\sum_{j=1}^{N}\int_0^{t}\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{j,N},M_{s-}^{j,N})\}}\,\mathcal{N}^j(du,ds)$$
$$\le M_0^{i,N}+\alpha\,J\left(C_1(t+\bar{X}_0^N)+Z_t^N\right),$$

and we find another positive constant C_1', depending only on C_1, α and J, such that inequality (16) holds.

□

A side result of the previous Lemma is the

Proof (Proof of Proposition 1) Let us notice that for any fixed *random environment*[1], we can construct explicitly a unique solution to the problem. Therefore there is a unique strong maximal solution $((X_t^{i,N},M_t^{i,N}))_{t\ge0}$ defined on a time interval of the type $[0,\tau^*)$, where τ^* is given by

$$\tau^*\;:=\;\inf\left\{t\ge0\;:\;\min(|(X_t^{1,N},M_t^{1,N})|,\ldots,|(X_t^{N,N},M_t^{N,N})|)=\infty\right\}.$$

But, inequalities (14) and (16) imply that a.s. the solutions to (3)-(4) are locally bounded, therefore $\tau^*=\infty$. In conclusion, the particle system (3)-(4) is *locally* strongly well-posed.

□

[1] A *random environment* is any realisation of the initial conditions, the Poisson processes and the delays at $t=0$ and frozen during the evolution of the dynamics.

Corollary 7 *There exist constants $C_M > 0$ such that if the initial laws m_0 and g_0 are compactly supported, then for any fixed $i \in \mathbb{N}$, it holds*

$$\mathbb{P}\Big(\sup_{[0,T]} |(X_t^{i,N}, M_t^{i,N})| \geq C_M \Big) \leq c_T \, e^{-C_T N}, \quad \forall T \geq 0,$$

for some c_T, C_T depending on the parameters of the system and the time horizon T.

Proof Let $i \in \mathbb{N}$ be fixed and consider $C(g_0)$ any upper bound of the support of g_0. From (14), we have that $X_t^{i,N} \leq C(g_0) + T$ for any $t \in [0,T]$. Moreover, using Markov's inequality, we have that

$$\mathbb{P}\big(Z_T^N \geq 2\delta_0 \, T \big) \; \leq \; e^{-2\delta_0 \, NT} \mathbb{E}\big[e^{N Z_T^N} \big],$$

but $Z_T^N = N^{-1} \sum_{j=1}^N \int_0^T \int_0^{\delta_0} \mathcal{N}^j(du, ds)$, is the mean of N i.i.d. Poisson random variables with parameter $(\delta_0 \, T)$, therefore

$$\mathbb{P}\big(Z_T^N \geq 2\delta_0 \, T \big) \; \leq \; e^{-\delta_0 \, NT(3-e)}.$$

If $C(m_0)$ stands for an upper bound of the support of m_0 then we conclude that

$$\mathbb{P}\Big(\sup_{[0,T]} M_t^{i,N} \geq C(m_0) + C_1' \big(T + C(g_0) + 2\delta_0 \, T \big) \Big) \xrightarrow{N \to \infty} 0, \quad \forall T \geq 0,$$

and the conclusion follows.

□

3 Path-wise uniqueness of the mean-field system

The object of this brief section is to prove under what circumstances the mean field equations (7)-(8) are likely to be well posed. Before that, we need to state some equivalent upper bounds on the solution to the limit equation (3)-(4) for the mean field system, that will be useful to prove the pathwise uniqueness result we are looking for. The proof of the next lemma is very similar to the arguments used to get (14) and (16) and therefore we do not go into full details.

Lemma 8 *Any solution $(M_t, X_t)_{t \geq 0}$ to (7)-(8) satisfies a.s.*

$$X_t \; \leq \; X_0 + t, \qquad \forall \, t \geq 0. \tag{18}$$

Moreover, there is a positive constant C_2, depending only on the parameters of the system, such that

$$\int_0^t \mathbb{E}[(1 + X_s) \, a(X_s, M_s)] \, ds \; \leq \; C_2(\mathbb{E}[X_0] + t), \qquad \forall \, t \geq 0. \tag{19}$$

As a consequence, there is another positive constant C_2', such that a.s.

$$M_t \; \leq \; M_0 + C_2'(t + \mathbb{E}[X_0]), \qquad \forall \, t \geq 0. \tag{20}$$

Proof Inequalities (18) is easily checked by recalling (7). Inequality (19) and (20), are a consequence of a change of variables. Indeed, we have that

$$\begin{aligned}\int_0^t \int_0^s \mathbb{E}[a(X_{s-w}, M_{s-w})]\, b(dw)\, ds &= \int_0^t b(dw) \int_w^t \mathbb{E}[a(X_{s-w}, M_{s-w})]\, ds \\ &\le \int_0^t b(dw) \int_0^t \mathbb{E}[a(X_s, M_s)]\, ds.\end{aligned}$$

Finally, we use again that $x \ge x_\delta(1 - \mathbf{1}_{\{x \le x_\delta\}})$ to get

$$\int_0^t \mathbb{E}[a(X_s, M_s)]\, ds \;\le\; x_\delta^{-1}(\mathbb{E}[X_0] + t) + \int_0^t \mathbb{E}[a(x_\delta, M_s)]\, ds,$$

the conclusion follows by recalling that $a(x_\delta, M_s) \le \delta_0$.

□

Proposition 9 *Path-wise uniqueness holds true for the mean field system (7)-(8), in the class of processes such that there exist deterministic locally bounded functions $A, B : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that a.s. (10) holds.*

Proof Let us consider two solutions $(X_t, M_t)_{t\ge 0}$ and $(X'_t, M'_t)_{t\ge 0}$, driven by the same Poisson process $(\mathcal{N}_t)_{t\ge 0}$ and identical initial conditions $(X_0, M_0) = (X'_0, M'_0)$. It holds

$$\begin{aligned}M_t - M'_t = -\alpha &\int_0^t (M_s - M'_s)\, ds \\ &+ \alpha\, J \int_0^t \int_0^s \mathbb{E}\big[a(X_{s-w}, M_{s-w}) - a(X'_{s-w}, M'_{s-w})\big]\, b(dw) ds\end{aligned}$$

then, using the arguments of Lemma 8, we have

$$\begin{aligned}\mathbb{E}\big[|M_t - M'_t|\big] \le \alpha &\int_0^t \mathbb{E}\big[|M_s - M'_s|\big]\, ds \\ &+\alpha\, J \int_0^t \int_0^s \mathbb{E}[|a(X_{s-w}, M_{s-w}) - a(X'_{s-w}, M'_{s-w})|]\, b(dw) ds \\ &\le \alpha \int_0^t \mathbb{E}\big[|M_s - M'_s|\big]\, ds + \alpha\, J \int_0^t \mathbb{E}\big[|a(X_s, M_s) - a(X'_s, M'_s)|\big]\, ds.\end{aligned}$$

For the $X_t - X'_t$ contribution we get similar bounds, indeed, we notice that

$$\begin{aligned}X_t - X'_t = -&\int_0^t X_{s-} \int_0^\infty \mathbf{1}_{\{u \le a(X_{s-}, M_{s-})\}} \mathcal{N}(du, ds) \\ &+ \int_0^t X'_{s-} \int_0^\infty \mathbf{1}_{\{u \le a(X'_{s-}, M'_{s-})\}} \mathcal{N}(du, ds),\end{aligned}$$

then, thanks to Itô's formula (using a regular version of $|\cdot|$ and then passing to the limit), we get

$$\mathbb{E}\big[|X_t - X'_t|\big] \leq \int_0^t \mathbb{E}\big[|X_s - X'_s|\, a(X_s, M_s) \wedge a(X'_s, M'_s)\big]\, ds$$
$$+ \int_0^t \mathbb{E}\big[|X_s \wedge X'_s|\, |a(X_s, M_s) - a(X'_s, M'_s)|\big]\, ds.$$

We know that a is a regular differentiable function, therefore it is Lipschitz continuous and bounded on compacts. Since both coordinates are bounded for some deterministic $(A(t), B(t))$ locally bounded functions, it follows that there exists C_T a constant depending only on an upper bound of A and B and the time horizon $T > 0$ such that

$$\mathbb{E}\big[|X_t - X'_t| + |M_t - M'_t|\big] \;\leq\; C_T \int_0^t \mathbb{E}\big[|X_s - X'_s| + |M_s - M'_s|\big]\, ds, \quad \forall\, 0 \leq t \leq T,$$

and the conclusion follows by applying Grönwall's lemma.

□

We can relax the previous assumptions by asking fast decay on the initial conditions. More precisely, if initial conditions are such that (13) holds, then thanks to Markov's inequality we have

$$\mathbb{P}\big(X_0 \geq R\big) \;\leq\; e^{-\omega R^{\xi}}\mathbb{E}\big[e^{\omega X_0^{\xi}}\big] \;\leq\; C_\omega\, e^{-\omega R^{\xi}},$$

for a positive ω and ξ given by (11). Moreover, thanks to (18), it follows that

$$\mathbb{P}(X_t \geq R) \;\leq\; \mathbb{P}\big(X_0 \geq R - t\big) \;\leq\; C_\omega\, e^{-\omega|R-t|^{\xi}} \leq C_\omega(t)\, e^{-\frac{\omega}{2}R^{\xi'}}.$$

for some constant $C_\omega(t)$ depending only on the time t and the coefficients of the system, and any $\xi' < \xi$. Moreover, since the exponential moments of X_0 exist, then any polynomial moment also does and $\mathbb{E}[X_0]$ is finite. We get an equivalent inequality for M_t:

$$\mathbb{P}(M_t \geq R) \;\leq\; \mathbb{P}\big(M_0 \geq R - C'_2(t + \mathbb{E}[X_0])\big) \;\leq\; C_\omega\, e^{-\omega(R - C'_2(t+\mathbb{E}[X_0]))^{\xi}}.$$

Proposition 10 *Under growth restriction (11) and mean field condition (12), path-wise uniqueness holds true for the mean field system (7)-(8), in the class of processes $(Y_t, M_t)_{t\geq 0}$ such that (13) holds true.*

Proof We start by noticing that since exponential moments are bounded, then all polynomial moments are bounded. In particular,

$$\mathbb{E}\big[a(X_t, M_t)^4\big] \leq \mathbb{E}\big[C_\xi^4(1 + X_t^{\xi-2} + M_t^{\xi-2})^4\big] < \infty,$$

for any time $t \geq 0$. Moreover, thanks to hypothesis (12), we have that

$$\big|a(X_s, M_s) - a(X'_s, M'_s)| \leq C_0\, a(X_s, M_s) \wedge a(X'_s, M'_s)\big(|X_s - X'_s\big| + |M_s - M'_s|\big).$$

Consider now

$$e(Y,M)=\Big\{\sup_{0\le s\le t}|(X_s,M_s)|\le R\Big\},$$

then, for any $R>0$, we have that

$$\begin{aligned}\mathbb{E}\big[|X_t-X_t'|\big] &\le C_0\,(1+R)\,a(R,R)\int_0^t \mathbb{E}\big[|X_s-X_s'|\big]\,ds + C_0\,Ra(R,R)\int_0^t \mathbb{E}\big[|M_s-M_s'|\big]\,ds\\ &+\int_0^t \mathbb{E}\big[(X_s-X_s')^4\big]^{1/4}\mathbb{E}\big[|a(X_s,M_s)\wedge a(X_s',M_s')|^4\big]^{1/4}\big]\\ &\qquad\times\mathbb{P}(e(X,M)^c)^{1/4}\mathbb{P}(e(X',M')^c)^{1/4}\,ds\\ &+\int_0^t \mathbb{E}\big[(X_s\wedge X_s')^4\big]^{1/4}\mathbb{E}\big[|a(X_s,M_s)-a(X_s',M_s')|^4\big]^{1/4}\big]\\ &\qquad\qquad\times\mathbb{P}(e(X,M)^c)^{1/4}\mathbb{P}(e(X',M')^c)^{1/4}\,ds,\end{aligned}$$

and that

$$\begin{aligned}\mathbb{E}\big[|M_t-M_t'|\big] &\le \alpha\,J\,C_0\,a(R,R)\int_0^t \mathbb{E}\big[|X_{s'}-X_{s'}'|\big]\,ds\\ &+\alpha\,(1+J\,C_0\,a(R,R))\int_0^t \mathbb{E}\big[|M_{s'}-M_{s'}'|\big]\,ds\\ &+\alpha\,J\int_0^t \mathbb{E}\big[|a(X_{s'},M_{s'})-a(X_{s'}',M_{s'}')|^2\big]^{1/2}\\ &\qquad\qquad\times\mathbb{P}(e(X,M)^c)^{1/4}\mathbb{P}(e(X',M')^c)^{1/4}\,ds.\end{aligned}$$

Using Gronwall's lemma, we conclude that there exists a constant C_T depending only on the parameters of the system, such that

$$\mathbb{E}\big[|X_t-X_t'|+|M_t-M_t'|\big]\;\le\;C_Te^{C_T Ra(R,R)}\,\mathbb{P}(e(X,M)^c)^{1/4}\mathbb{P}(e(X',M')^c)^{1/4},$$

but using the fast decay at infinite

$$\mathbb{E}\big[|X_s-X_s'|+|M_s-M_s'|\big]\;\le\;C_Te^{C_T Ra(R,R)}e^{-\frac{\omega}{4}R^{\xi-1}},$$

where the constants depends on the time horizon T, the coefficients of the system, but not on R. Finally thanks to hypothesis (11), we get that

$$\exp\Big(C_T\,R\,a(R,R)-\frac{\omega}{4}R^{\xi-\frac{1}{2}}\Big)\xrightarrow{R\to\infty}0,$$

and the conclusion follows.

□

4 Consistency of the mean field system

So far, we know that, for each $N \geq 1$ and $(g_0 \otimes m_0)$-*chaotic initial states*, there exists a unique solution to (3)-(4). We now study the convergence of the this set of solutions as N goes to infinity, i.e. the tightness of the sequence of empirical measures μ_N. To that aim, we start by recalling (see e.g. Jacob-Shiryaev [7, Theorem 4.5, page 356]):

Aldous tightness criterion: *the sequence of adapted processes* $(X_t^{1,N}, M_t^N)$ *is tight if*

1. *for all* $T > 0$*, all* $\epsilon > 0$*, it holds*

$$\lim_{\delta \to 0+} \limsup_{N \to \infty} \sup_{(S,S') \in A_{\delta,T}} \mathbb{P}\big(|M_S^{1,N} - M_{S'}^{1,N}| + |X_S^{1,N} - X_{S'}^{1,N}| > \epsilon\big) = 0;$$

where $A_{\delta,T}$ *is the set of stopping times* (S, S') *such that* $0 \leq S \leq S' \leq S + \delta \leq T$ *a.s., and*

2. *for all* $T > 0$*,*

$$\lim_{K \to \infty} \sup_{N \geq 1} \mathbb{P}\Big(\sup_{t \in [0,T]} (M_t^{1,N} + X_t^{1,N}) \geq K\Big) = 0.$$

Proposition 11 *Assume hypothesis* (5) *and* (6)*. Consider two probability distributions* g_0, m_0 *compactly supported or such that* (13) *holds, and the correspondent family of solutions to* (3)-(4) *indexed by* N*, starting with some i.i.d. random variables* $(X_0^{i,N}, M_0^{i,N})$ *with common law* $g_0 \otimes m_0$*. Then*

(i) the sequence of processes $(X_t^{1,N}, M_t^{1,N})_{t \geq 0}$ *is tight in* $\mathbb{D}(\mathbb{R}_+^2)$*;*
(ii) the sequence of empirical measures μ_N *is tight in* $\mathbb{P}\big(\mathbb{D}(\mathbb{R}_+^2)\big)$*.*

Let us remark that the sequence $Z^i := (X_t^{i,N}, M_t^{i,N})$ is exchangeable, then (ii) follows from (i) thanks to Sznitman [18, Proposition 2.2-(ii)]).

Proof We only need to show the Aldous tightness criterion. We start by the second condition, from estimate (14) we get that

$$\mathbb{E}\big[\sup_{0 \leq s \leq T} X_s^{1,N}\big] \leq \mathbb{E}[X_0^{1,N}] + T < \infty,$$

and recalling (16), we notice that

$$\mathbb{E}\big[\sup_{t \in [0,T]} M_t^{1,N}\big] \leq \mathbb{E}[M_0^{1,N}] + C_1'(T + \mathbb{E}[\bar{X}_0^N] + \mathbb{E}[Z_T^N]) < \infty,$$

because Z_T^N is the mean of N i.i.d Poisson$(\delta_0 T)$-distributed random variables. We deduce that the expectation of the lefthand side is finite independently of the value of N, and the conclusion follows.

To prove the first point, we notice that by definition

$$|X_S^{1,N} - X_{S'}^{1,N}| \leq (S' - S) + \int_S^{S'} \int_0^\infty X_{s-}^{1,N} \mathbf{1}_{\{u \leq a(X_{s-}^{1,N}, M_{s-}^N)\}} \mathcal{N}^1(du, ds),$$

and

$$|M_S^{1,N} - M_{S'}^{1,N}| \le \frac{\alpha\,\varepsilon}{N} \sum_{j\neq i} \int_{S-\tau_j}^{S'-\tau_j} \int_0^\infty \mathbf{1}_{u \le a(X_{s-}^{j,N}, M_{s-}^{j,N})} \mathcal{N}^j(du, ds) \\ + \alpha \int_S^{S'} M_s^{1,N}\, ds.$$

Moreover, using Markov's inequality

$$\mathbb{P}\Big(\int_S^{S'} \int_0^\infty X_{s-}^{1,N} \mathbf{1}_{\{u \le a(X_{s-}^{1,N}, M_{s-}^N)\}} \mathcal{N}^1(du, ds) > 0 \Big) \\ \le \mathbb{E}\Big[\int_S^{S+\delta} a(X_s^{1,N}, M_s^N)\, ds \Big] \\ \le \delta^{1/2} \times \mathbb{E}\Big[\Big(\int_0^T a(X_s^{1,N}, M_s^N)^2\, ds \Big)^{1/2} \Big],$$

which is finite thanks to the initial conditions. All other terms can be handled in a similar way, and the conclusion follows.

□

The natural next step, in the proof of existence of solutions to the non-linear SDE, is to show that any limit point of the tight sequence μ_N is a solution of the mean field limit system, which is usually called consistency of the particle system. This result is stated in the following

Proposition 12 *Under the same hypotheses of Proposition 11, any limit point μ of μ_N a.s. belongs to*

$$\mathcal{S} := \Big\{ \mathcal{L}((X_t, M_t)_{t\ge 0}) \,:\, (X_t, M_t)_{t\ge 0} \text{ is a solution to (7)-(8) such that} \\ \mathcal{L}((X_0, M_0)) = g_0 \otimes m_0 \text{ and such that for any } t \ge 0 \\ \int_0^t \int_0^s \mathbb{E}\big[a(X_{s-w}, M_{s-w})\, b(dw) \big]\, ds < \infty \Big\}.$$

A previous step that simplifies the proof of this result is the

Lemma 13 *Let us consider $t \ge 0$ fixed and define $\pi_t : \mathbb{D}(\mathbb{R}_+^2) \to \mathbb{R}_+^2$, by*

$$\pi_t(\gamma, \beta) = (\gamma_t, \beta_t).$$

Then, $Q \in \mathbb{P}(\mathbb{D}(\mathbb{R}_+^2))$ belongs to $\mathcal{S}$ if the following conditions are satisfied:

(a) $Q \circ \pi_0^{-1} = (g_0 \otimes m_0)$;
(b) for all $t \ge 0$,

$$\int_{\mathbb{D}(\mathbb{R}_+^2)} \int_0^t \int_0^s a(\gamma_{s-w}, \beta_{s-w})\, b(dw)\, ds\, Q(d\gamma, d\beta) < \infty;$$

(c) for any $0 \le s_1 < \ldots < s_k < s < t$, any $\varphi_1, \ldots, \varphi_k \in C_b(\mathbb{R}_+^2)$, and any $\varphi \in C_b^2(\mathbb{R}_+^2)$, it holds

$$
\begin{aligned}
F(Q) := \int_{\mathbb{D}(\mathbb{R}_+^2)} \int_{\mathbb{D}(\mathbb{R}_+^2)} & Q(d\gamma^1, d\beta^1)\, Q(d\gamma^2, d\beta^2)\, \varphi_1(\gamma^1_{s_1}, \beta^1_{s_1}) \ldots \varphi_k(\gamma^1_{s_k}, \beta^1_{s_k}) \\
& \Big[\varphi(\gamma^1_t, \beta^1_t) - \varphi(\gamma^1_s, \beta^1_s) \\
& - \int_s^t \partial_\beta \varphi(\gamma^1_{s'}, \beta^1_{s'}) \big[- \alpha \beta^1_{s'} + \alpha\, J \int_0^{s'} a(\gamma^2_{s'-w}, \beta^2_{s'-w})\, b(dw) \big]\, ds' \\
& - \int_s^t \partial_\gamma \varphi(\gamma^1_{s'}, \beta^1_{s'})\, ds' - \int_s^t a(\gamma^1_{s'}, \beta^1_{s'}) \big[\varphi(0, \beta^1_{s'}) - \varphi(\gamma^1_{s'}, \beta^1_{s'})\big]\, ds' \Big] = 0.
\end{aligned}
$$

Proof Let us consider a process $(X_t, M_t)_{t \ge 0}$ of law Q which satisfies (a), (b) and (c). From (a) and the independency of m_0 and g_0, we have

$$
\mathcal{L}((X_0, M_0)) \;=\; g_0 \otimes m_0,
$$

and from (b) we have that

$$
\int_0^t \int_0^s \mathbb{E}\big[a(X_{s-w}, M_{s-w})\big]\, b(dw)\, ds < \infty, \quad \forall\, t \ge 0.
$$

Finally, from (c) we have that for any $\varphi \in C_b^2(\mathbb{R}_+^2)$, the process

$$
\begin{aligned}
& \varphi(X_t, M_t) - \varphi(X_0, M_0) \\
& \quad - \int_0^t \partial_\beta \varphi(X_s, M_s) \Big[- \alpha M_s + \alpha\, J \int_0^s \mathbb{E}[a(X_{s-s'}, M_{s-s'})]\, b(ds') \Big] ds \\
& \qquad - \int_0^t \partial_\gamma \varphi(X_s, M_s)\, ds - \int_0^t a(X_s, M_s) \big(\varphi(0, M_s) - \varphi(X_s, M_s)\big) ds,
\end{aligned}
$$

is a local martingale. The conclusion follows as an application of Jacob-Shiryaev [7, Theorem II.2.42 page 86] and [7, Theorem III.2.26 page 157]. This result is classic, but for completeness of the present manuscript, we provide some remarks on Appendix A.

□

We finish this section by giving the proof of Proposition 12. To that aim, let us recall some general results of stochastic calculus for jump processes. Let $\varphi \in C^1(\mathbb{R}_+^2)$ a regular test function, the Itô's formula writes

$$
\begin{aligned}
\varphi(X^{i,N}_t, M^{i,N}_t) \;=\; & \mathcal{M}^{i,N}_\varphi(t) + \varphi(X^{i,N}_0, M^{i,N}_0) \\
& + \int_0^t \partial_x \varphi(X^{i,N}_s, M^{i,N}_s)\, ds - \alpha \int_0^t M^{i,N}_s \partial_m \varphi(X^{i,N}_s, M^{i,N}_s)\, ds \\
& + \int_0^t \big[\varphi\big(0, M^{i,N}_s\big) - \varphi(X^{i,N}_{s-}, M^{i,N}_s)\big] a(X^{i,N}_s, M^{i,N}_s)\, ds \\
+ \sum_{j \ne i} \int_0^{t-\tau_j} & \Big[\varphi\big(X^{i,N}_s, M^{i,N}_{s-} + \frac{\alpha J}{N}\big) - \varphi(X^{i,N}_s, M^{i,N}_{s-})\Big] a(X^{j,N}_s, M^{j,N}_s)\, ds.
\end{aligned} \tag{21}
$$

where the respective local martingale $(\mathcal{M}_\varphi^{i,N}(s))$ is defined by

$$\begin{aligned}\mathcal{M}_\varphi^{i,N}(t) &:= \int_0^t \big[\varphi(0, M_{s-}^{i,N}) - \varphi(X_{s-}^{i,N}, M_{s-}^{i,N})\big]\\ &\quad\times \Big[\int_0^\infty \mathbf{1}_{\{u \le a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \mathcal{N}^i(du, ds) - a(X_s^{i,N}, M_s^{i,N})\, ds\Big]\\ &\quad+ \sum_{j\neq i} \int_0^{t-\tau_j} \Big[\varphi\big(X_{s_-}^{i,N}, M_{s_-}^{i,N} + \frac{\alpha J}{N}\big) - \varphi(X_{s_-}^{i,N}, M_{s_-}^{i,N})\Big]\\ &\quad\times \Big[\int_0^\infty \mathbf{1}_{\{u \le a(X_{s-}^{j,N}, M_{s-}^{j,N})\}} \mathcal{N}^j(du, ds) - a(X_s^{j,N}, M_s^{j,N})\, ds\Big]. \end{aligned} \tag{22}$$

Proof (Proof of Proposition 12) At this point, the problem is reduced to prove that conditions (a), (b) and (c) of Lemma 13 hold. Since we do not have much information about μ we cannot work directly with it. On the other hand, we know that μ_N (up to subsequence) is converging to μ, therefore, it seems natural to use equations (3)-(4) adequately and then pass to the limit.

Step 1. Let us recall that for any $N \ge 1$, the random variables $X_0^{i,N}$ are i.i.d. with common law g_0, and that the random variables $M_0^{i,N}$ are also i.i.d. with common law m_0. It follows that

$$\mu \circ \pi_0^{-1} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \delta_{(X_0^{i,N}, M_0^N)} = (g_0 \otimes m_0).$$

We also have, by the Fatou's lemma and inequality (15), that

$$\begin{aligned}&\mathbb{E}\Big[\int_{\mathbb{D}(\mathbb{R}_+^2)} \int_0^t \int_0^s \big[a(\gamma_{s-w}, \beta_{s-w}) \wedge K\big]\, b(dw)\, ds\, \mu(d\gamma, d\beta)\Big]\\ &\qquad\le \liminf_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \int_0^t \int_0^s \mathbb{E}[a(X_{s-w}^{i,N}, M_{s-w}^N)]\, b(dw)\, ds,\end{aligned}$$

is finite for any $t \ge 0$. Letting $K \to \infty$ we get (b).

Step 2. It only remains to prove (c), to that aim, we start by noticing that $F(\mu_N)$ writes

$$
\begin{aligned}
F(\mu_N) \;=\; & \frac{1}{N}\sum_{i=1}^{N}\varphi_1(X_{s_1}^{i,N},M_{s_1}^{i,N})\dots\varphi_k(X_{s_k}^{i,N},M_{s_k}^{i,N}) \\
& \Big[\varphi(X_t^{i,N},M_t^{i,N})-\varphi(X_s^{i,N},M_s^{i,N}) \\
& -\int_s^t \partial_\gamma\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\,ds' + \alpha\int_s^t M_{s'}^{i,N}\partial_\beta\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\,ds' \\
& -\int_s^t \big[\varphi(0,M_{s'}^{i,N})-\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\big]a(X_{s'}^{i,N},M_{s'}^{i,N})\,ds' \\
& -\int_s^t \partial_\beta\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\frac{\alpha\, J}{N}\sum_{j=1}^{N}\int_0^{s'} a(X_{s'-w}^{j,N},M_{s'-w}^{j,N})\,b(dw)\,ds'\Big].
\end{aligned}
$$

At the same time, using the Itô's formula (21) to the test function $\varphi(\cdot,\cdot)$, we have

$$
\begin{aligned}
\varphi(X_t^{i,N},M_t^{i,N}) \;=\; & \varphi(X_0^{i,N},M_0^{i,N})+\int_0^t \partial_\gamma\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\,ds' \\
& -\alpha\int_0^t M_{s'}^{i,N}\partial_\beta\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\,ds' \\
& +\int_0^t \big[\varphi(0,M_{s'}^{i,N})-\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\big]\int_0^\infty \mathbf{1}_{\{u\le a(X_{s'}^{i,N},M_{s'}^{i,N})\}}\mathcal{N}^i(du,ds') \\
& +\sum_{j\neq i}\int_0^{t-\tau_j}\big[\varphi(X_{s'}^{i,N},M_{s'}^{i,N}+\frac{\alpha\, J}{N})-\varphi(X_{s'}^{i,N},M_{s'}^{i,N})\big] \\
& \times\int_0^\infty \mathbf{1}\{u\le a(X_{s'}^{j,N},M_{s'}^{j,N})\}\mathcal{N}^j(du,ds'),
\end{aligned}
$$

implying, that $F(\mu_N)$ can be rewritten by

$$
\begin{aligned}
F(\mu_N) = & \frac{1}{N}\sum_{i=1}^{N}\varphi_1(X_{s_1}^{i,N},M_{s_1}^{i,N})\dots\varphi_k(X_{s_k}^{i,N},M_{s_k}^{i,N}) \\
& \big[(R_t^{i,N}-R_s^{i,N})+(\varDelta_t^{i,N}-\varDelta_s^{i,N})\big],
\end{aligned}
$$

where

$$
\begin{aligned}
R_t^{i,N} := & \int_0^t \big[\varphi(0,M_{s-}^{i,N})-\varphi(X_{s-}^{i,N},M_{s-}^{i,N})\big] \\
& \times\Big[\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{i,N},M_{s-}^{i,N})\}}\mathcal{N}^i(du,ds)-a(X_{s-}^{i,N},M_{s-}^{i,N})\,ds\Big],
\end{aligned}
$$

and

$$\begin{aligned}\varDelta_t^{i,N} := \sum_{j\neq i}\int_0^{t-\tau_j}\int_0^\infty & \Big[\varphi\Big(X_{s-}^{i,N}, M_{s-}^{i,N}+\frac{\alpha\, J}{N}\Big) \\ & - \varphi(X_{s-}^{i,N}, M_{s-}^{i,N})\Big]\mathbf{1}_{\{u\le a(X_{s-}^{j,N}, M_{s-}^{j,N})\}}\mathcal{N}^j(du,ds) \\ & - \int_0^t \partial_\beta\varphi(X_s^{i,N}, M_s^{i,N})\frac{\alpha\, J}{N}\sum_{j=1}^N\int_0^s a(X_{s-w}^{j,N}, M_{s-w}^{j,N})\, b(dw)\, ds.\end{aligned}$$

Using that the Poisson processes $\mathcal{N}^i$ are i.i.d., we get that the compensated martingales $R_t^{i,N}$ are orthogonal, and thanks to the exchangeability, we get that

$$\mathbb{E}[|F(\mu_N)|] \;\le\; \frac{C_F}{\sqrt{N}}\mathbb{E}[(R_t^{1,N}-R_s^{1,N})^2]^{1/2} + C_F\,\mathbb{E}[|\varDelta_t^{1,N}| + |\varDelta_s^{1,N}|]$$

for some positive C_F depending on the upper bounds of the test functions composing F. Moreover, the first expectation is bounded uniformly on N:

$$\mathbb{E}\big[(R_t^{1,N}-R_s^{1,N})^2\big] \;\le\; C_F\int_0^t \mathbb{E}\big[a(X_s^{1,N}, M_s^N)\big]\, ds,$$

which is finite thanks to (15).

For the second expectation, we split $\varDelta_t^{1,N}$ in four quantities that can be handled separately:

$$\begin{aligned}|\varDelta_t^{1,N}| \le \int_0^{t-\tau_1} & \Big|\varphi(X_{s-}^{1,N}, M_{s-}^{1,N}+\frac{\alpha\, J}{N}) \\ & - \varphi(X_{s-}^{1,N}, M_{s-}^{1,N})\Big|\mathbf{1}_{\{u\le a(X_{s-}^{1,N}, M_{s-}^{1,N})\}}\mathcal{N}^1(du,ds) \\ & + \Big|\sum_{j=1}^N\int_0^{t-\tau_j}\big[\varphi(X_{s-}^{1,N}, M_{s-}^{1,N}+\frac{\alpha\, J}{N}) - \varphi(X_{s-}^{1,N}, M_{s-}^{1,N})\big] \\ & \qquad \times\Big[\int_0^\infty \mathbf{1}_{\{u\le a(X_{s-}^{j,N}, M_{s-}^{j,N})\}}\mathcal{N}^j(du,ds) - a(X_s^{j,N}, M_s^{j,N})\, ds\Big]\Big| \\ & + \sum_{j=1}^N\int_0^{t-\tau_j}\Big|\big[\varphi(X_{s-}^{1,N}, M_{s-}^{1,N}+\frac{\alpha\, J}{N}) - \varphi(X_{s-}^{1,N}, M_{s-}^{1,N}) \\ & \qquad - \frac{\alpha J}{N}\partial_\beta\varphi(X_{s-}^{1,N}, M_{s-}^{1,N})\big]a(X_s^{j,N}, M_s^{j,N})\Big|\, ds \\ & \qquad + C_F\frac{\alpha\, J}{N}\Big|\sum_{j=1}^N\Big(\int_0^{t-\tau_j} a(X_s^{j,N}, M_s^{j,N})\, ds \\ & \qquad\quad - \int_0^t\int_0^s a(X_{s-w}^{j,N}, M_{s-w}^{j,N})\, b(dw)\, ds\Big)\Big| \\ & \qquad\qquad := \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 + \mathscr{T}_4.\end{aligned}$$

The first three terms are controllable simply using that $\varphi \in C_b^2(\mathbb{R}_+^2)$. Indeed, for any $(x,m) \in \mathbb{R}_+^2$, we have that

$$\varphi\Big(x, m+\frac{\alpha J}{N}\Big) \;=\; \varphi(x,m)+\frac{\alpha J}{N}\,\partial_\beta\varphi(x,m)+\frac{1}{2}\Big(\frac{\alpha J}{N}\Big)^2\,\partial^2_{\beta\beta}\varphi(x,m)+O(N^{-2}),$$

using again (15) we get that the respective expectations are going to 0 when N goes to infinity (using Holder's inequality to find the convergence).

The contribution of $\mathscr{T}_4$ must be handled more carefully. Nevertheless, we have that

$$\begin{aligned}\mathbb{E}_{\text{delays}}\Big[\int_0^{t-\tau_j} a(X_s^{j,N},M_s^{j,N})\,ds\Big] \;&=\; \int_0^\infty\int_0^{t-w} a(X_s^{j,N},M_s^{j,N})\,ds\,b(dw)\\ &=\; \int_0^t\int_0^s a(X_{s-w}^{j,N},M_{s-w}^{j,N})\,b(dw)\,ds.\end{aligned}$$

for any $1 \le j \le N$, therefore, each term defined by

$$\mathscr{T}_{j,4}(t) \;:=\; \int_0^{t-\tau_j} a(X_s^{j,N},M_s^{j,N})\,ds - \int_0^t\int_0^s a(X_{s-w}^{j,N},M_{s-w}^{j,N})\,b(dw)\,ds,$$

has zero expectation. Using that the delays τ_j are i.i.d, we get that

$$\mathbb{E}\big[\mathscr{T}_{j,4}(t)\times\mathscr{T}_{k,4}(t)\big] \;=\; \mathbb{E}\big[\mathbb{E}_{\text{delays}}[\mathscr{T}_{j,4}(s)]\,\times\,\mathbb{E}_{\text{delays}}[\mathscr{T}_{k,4}(s)]\big]=0,$$

if $j \neq k$, then

$$\begin{aligned}\mathbb{E}[\mathscr{T}_4] \;\le\; C_F\,\frac{\alpha\,J}{N}\Big(\sum_{j,k=1}^N \mathbb{E}\big[\mathscr{T}_{j,4}(t)\times\mathscr{T}_{k,4}(t)\big]\Big)^{1/2}\,ds\\ =\; C_F\,\frac{\alpha\,J}{\sqrt{N}}\Big(\mathbb{E}\big[(\mathscr{T}_{1,4}(t))^2\big]\Big)^{1/2}.\end{aligned}$$

Finally, we see that

$$\begin{aligned}\mathbb{E}\big[\mathscr{T}_{1,4}(s)^2\big] \;\le\; 2\,\mathbb{E}\Big[\Big(\int_0^{t-\tau_1} a(X_s^{1,N},M_s^{1,N})\,ds\Big)^2\\ +\Big(\int_0^t\int_0^s a(X_{s-w}^{1,N},M_{s-w}^{1,N})\,b(dw)\,ds\Big)^2\Big],\end{aligned}$$

and the righthand side is upper bounded independently of N

Step 3. Before passing to the limit we still need to be sure that no mass is added in the discontinuity points of the paths, i.e. we need to check that for any $t \ge 0$, a.s.,

$$\mu(\{(\gamma,\beta)\;:\;\Delta(\gamma,\beta)(t)\neq 0\})=0.$$

The proof is exactly as in [5, Theorem 5-(iii)-Part 2] but for completeness we give some remarks. In order to get a contradiction, we assume that there are some $b,d>0$ such that

$$\mathbb{P}[E]>0,\quad\text{with}\quad E:=\big\{\mu(\{(\gamma,\beta):\max(|\Delta\gamma(t)|,|\Delta\beta(t)|)>b\})>d\big\}.$$

Therefore, for any $\epsilon > 0$, it holds

$$E \subset \{\mu(B_b^\epsilon) > d\}, \quad B_b^\epsilon := \{\mu : \sup_{s\in(t-\epsilon,t+\epsilon)} \max(|\Delta\gamma^1(s)|, |\Delta\gamma^2(s)|) > b\}.$$

Moreover, B_b^ϵ is an open subset of $\mathbb{D}(\mathbb{R}_+^2)$, then

$$\mathcal{P}_{b,d}^\epsilon := \{Q \in \mathbb{P}((\mathbb{R}_+^2) : Q(B_b^\epsilon) > d\} \subset \mathbb{P}(\mathbb{D}(\mathbb{R}_+^2)$$

is also an open set. Thanks of Portmanteau theorem we get that for any $\epsilon > 0$,

$$\liminf_{N\to\infty} \mathbb{P}(\mu_N \in \mathcal{P}_{b,d}^\epsilon) \geq \mathbb{P}(\mu \in \mathcal{P}_{b,d}^\epsilon) \geq \mathbb{P}(E) > 0.$$

On the other hand, for N large enough, the jumps in equation (4) are smaller than b and then the problem is reduced to control the size of the jumps in equation (3), and in particular to show that

$$\mathbb{P}(\mu_N \in \mathcal{P}_{b,d}^\epsilon) \leq \mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\left\{\int_{t-\epsilon}^{t+\epsilon} \mathbf{1}_{\{u\leq a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \mathcal{N}^i(du,ds)\geq 1\right\}} \geq b\Big) \to 0,$$

which can be easily done using the same arguments of the proof of Proposition 11.

Step 4. Now we see that F is a continuous function at any point $Q \in \mathbb{P}(\mathbb{D}(\mathbb{R}_+^2)$ such that

$$Q(\{(\gamma,\beta) \,:\, \Delta(\gamma,\beta)(s_1) = \ldots = \Delta(\gamma,\beta)(s_k) = \Delta(\gamma,\beta)(s) = \Delta(\gamma,\beta)(t) = 0\}) \;=\; 1,$$

and

$$\int_{\mathbb{D}(\mathbb{R}_+^2)} \int_0^t \int_0^s a(\gamma_{s-w}, \beta_{s-w})\, b(dw)\, ds\, Q(d\gamma, d\beta) \;<\; \infty.$$

Thanks to Step 2 and 3 we know that our limit belongs to this subset of $\mathbb{P}(\mathbb{D}(\mathbb{R}_+^2))$, and therefore

$$\mathbb{E}\big[|F(\mu)|\big] \;\leq\; \lim_{K\to\infty} \limsup_{N\to\infty} \mathbb{E}\big[|F(\mu_N)| \wedge K\big] \;=\; 0.$$

$\square$

5 Uniqueness of the weak solutions

We start this section by proving the existence and uniqueness for the nonlinear mean-field system (7)-(8). To that aim, we will use that if if the initial law $(g_0 \otimes m_0)$ is compactly supported, then thanks to inequalities (18) and (20), we get that a.s.

$$X_t \;\leq\; X_0 + t \;\leq\; C(g_0) + t =: A(t),$$

and

$$M_t \;\leq\; M_0 + C_2'(t + \mathbb{E}[Y_0]) \;\leq\; C(m_0) + C_2'(t + \mathbb{E}[Y_0]) =: B(t),$$

with $C(g_0)$ (respectively $C(m_0)$) any upper bound of the support of g_0 (respectively m_0).

Proof (Proof of Theorem 3) Proposition 12 gave us the existence in law of the solutions, such that (9) holds true. Furthermore, Proposition 9 implies that there is unique path-wise of the solutions in the compactly supported case, and this solutions is exactly a process with law μ (it suffices to pass to the limit in the associated particle system).

□

The existence of a strong solution in the case of exponential decay of initial conditions (ie. Theorem 4) is similar and therefore omitted.

So far, we have proved that the particle system is consistent with the mean field representation, and that there are weak solutions to (7)-(8), but to prove the convergence of the empirical mean towards the solution to the mean field limit equation, it is necessary to prove first that the set of solutions in the space of compactly supported or exponential decay of the initial data is a single element. This result is stated in the following:

Lemma 14 *Consider $f_t(x,m)$ and $g_t(x,m)$ be two compactly supported functions for any $t \geq 0$ or with exponential decay at infinite in the sense of (13). Assume furthermore that for all $\varphi \in C_c^2(\mathbb{R}_+^2)$ and that for all $t \geq 0$ it holds*

$$
\begin{aligned}
\int_{\mathbb{R}_+^2} \varphi(x,m) h_t(dx,dm) &= \int_{\mathbb{R}_+^2} \varphi(x,m) h_0(dx,dm) \\
&\quad + \int_0^t \int_{\mathbb{R}_+^2} \partial_x \varphi(x,m) h_s(dx,dm)\, ds \\
-\alpha \int_0^t \int_{\mathbb{R}_+^2} \Big[m - J \int_0^s \int_{\mathbb{R}_+^2} & a(y,r) f_{s-w}(dy,dr) b(dw)\Big] \partial_m \varphi(x,m) h_s(dx,dm)\, ds \\
&\quad + \int_0^t \int_{\mathbb{R}_+^2} \big(\varphi(0,m) - \varphi(x,m)\big) a(x,m) h_s(dx,dm)\, ds, \qquad (23)
\end{aligned}
$$

for $h_t = f_t$ or $h_t = g_t$, then $g_t = f_t$.

Proof For simplicity we divide the proof in 4 steps.
Step 0. Let us fix some $\varphi \in C_c^2(\mathbb{R}_+^2)$ and $t \geq 0$, and set

$$
\begin{aligned}
\mathcal{A}_t \varphi(x,m) &= \partial_x \varphi(x,m) \\
&\quad - \alpha \Big[m - J \int_0^t \int_{\mathbb{R}_+^2} a(y,r) f_{t-w}(dy,dr) b(dw)\Big] \partial_m \varphi(x,m) \\
&\quad + \big(\varphi(0,m) - \varphi(x,m)\big) a(x,m).
\end{aligned}
$$

If we prove that for any $\mu \in \mathbb{P}(\mathbb{R}_+^2)$ compactly supported (respectively with exponential decay at infinite), there exists at most one h in $L^\infty_{loc}([0,\infty), \mathbb{P}(\mathbb{R}_+^2))$, compactly supported for any time (respectively exponentially decaying), such that for all $t \geq 0$, and $\varphi \in C_c^2(\mathbb{R}_+^2)$,

$$\int_{\mathbb{R}_+^2} \varphi(x,m) h_t(dx,dm) = \int_{\mathbb{R}_+^2} \varphi(x,m)\mu(dx,dm) \\ + \int_0^t \int_{\mathbb{R}_+^2} \mathcal{A}_s \varphi(x,m) h_s(dx,dm)\, ds, \quad (24)$$

then we will conclude the proof since f and g solve the equation with $\mu = f_0$.

Let us recall that for any $\mu \in \mathbb{P}(\mathbb{R}_+^2)$, a continuous adapted $\mathbb{R}_+^2$-valued process $(\mathcal{X}_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ is said to solve the martingale problem $MP((\mathcal{A}_t)_{t\geq 0}, \mu)$ if $\mathcal{P} \circ \mathcal{X}_0^{-1} = \mu$ and if for any $\varphi \in C_c^2(\mathbb{R}_+^2)$, $(M_t^\varphi)_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, where

$$M_t^\varphi = \varphi(\mathcal{X}_t) - \int_0^t \mathcal{A}_s \varphi(\mathcal{X}_s)\, ds.$$

Moreover, using [1, Theorem 5.2, Remark 3.1] uniqueness for (24) holds if

(i) there exists a countable set $(\varphi_k)_{k\geq 1} \subset C_c^2(\mathbb{R}_+^2)$ such that for all $t \geq 0$, $\{(\varphi, \mathcal{A}_t\varphi), \varphi \in C_c^2(\mathbb{R}_+^2)\}$ is contained in the closure for the bounded point wise convergence of $\{(\varphi_k, \mathcal{A}_t \varphi_k), k \geq 1\}$,
(ii) for each $(x_0, m_0) \in \mathbb{R}_+^2$, there exists a solution to $MP((\mathcal{A}_t)_{t\geq 0}, \delta_{(x_0,m_0)})$,
(iii) for each $(x_0, m_0) \in \mathbb{R}_+^2$, uniqueness in law holds for $MP((\mathcal{A}_t)_{t\geq 0}, \delta_{(x_0,m_0)})$.

The rest of the proof is devoted to prove that the previous conditions hold.
Step 1. Let us take a countable set $(\psi_k)_{k\geq 1}$ dense in $C_c^2(\mathbb{R}_+^2)$, i.e, such that for any $\psi \in C_c^2$, there exists a subsequence (ψ_{k_n}) such that

$$\lim_{n\to\infty} \left(\|\psi_{k_n} - \psi\|_\infty + \|\nabla_{x,m}\psi_{k_n} - \nabla_{x,m}\psi\|_\infty + \|D^2_{x,m}\psi_{k_n} - D^2_{x,m}\psi\|_\infty \right) = 0.$$

Let us notice that for any (x,m) fixed, it holds

$$|\mathcal{A}_t\psi_{k_n}(x,m) - \mathcal{A}_t\psi(x,m)| \leq \|\partial_x \psi_{k_n} - \partial_x\psi\|_\infty + 2\|\psi_{k_n} - \psi\|_\infty a(x,m) \\ + \alpha \|\partial_m \psi_{k_n} - \partial_m \psi\|_\infty \Big[m + J \int_0^t \int_{\mathbb{R}_+^2} a(y,r) f_{t-w}(dy,dr) b(dw) \Big] \xrightarrow{n\to\infty} 0.$$

Secondly, we notice that since ψ_{k_n} and its derivatives up to second order are going to ψ in norm L^∞, then necessarily the support of ψ_{k_n} is going to the support of ψ. Therefore, there is a constant C depending on the support of ψ such that for any n large enough

$$|\mathcal{A}_t\psi_{k_n}(x,m)| \leq \|\partial_x\psi_{k_n}\|_\infty + 2C\|\psi_{k_n}\|_\infty \\ + \alpha\|\partial_m\psi_{k_n}\|_\infty \Big[C + J\int_0^t \int_{\mathbb{R}_+^2} a(y,r) f_{t-w}(dy,dr) b(dw) \Big],$$

using that f_t is compactly supported (or in the exponentially decaying case using (9)), we take $\sup_n$ and conclude (i).

Step 2. We observe that a process $(X_t, M_t)_{t\geq 0}$ is a solution to the martingale problem $MP((\mathcal{A}_t)_{t\geq 0}, \delta_{(x_0,m_0)})$ if and only if there exists a $(\mathcal{F}_t)_{t\geq 0}$-Poisson process $(\mathcal{N}_t)_{t\geq 0}$ such that

$$X_t = x_0 + t - \int_0^t X_{s-} \int_0^\infty \mathbf{1}_{\{u\leq a(X_{s-},M_{s-})\}} \mathcal{N}(du, ds), \tag{25}$$

and

$$M_t = m_0 - \alpha\Big[\int_0^t M_s\, ds - J\int_0^t \int_0^s \int_{\mathbb{R}_+^2} a(y,r)\, f_{s-w}(dy, dr)b(dw)ds\Big]. \tag{26}$$

Moreover, the proof of path wise uniqueness for (25)-(26) is identical with the proof of uniqueness of (7)-(8) and left it to the reader.

Step 3. It remains to prove the existence of solutions to the martingale problem to conclude. To that aim, we can use a Picard iteration argument. Indeed, consider the constant process $(X_t^0, M_t^0) = (x_0, m_0)$ and define recursively

$$X_t^{n+1} = x_0 + t - \int_0^t X_{s-}^n \int_0^\infty \mathbf{1}_{\{u\leq a(X_{s-}^n,M_{s-}^n)\}} \mathcal{N}(du, ds)$$

and

$$M_t^{n+1} = m_0 - \alpha\Big[\int_0^t M_s^n\, ds - J\int_0^t \int_0^s \int_{\mathbb{R}_+^2} a(y,r)\, f_{s-w}(dy, dr)b(dw)ds\Big].$$

We first notice that

$$\mathbb{E}\big[\sup_{[0,T]} |M_t^{n+1} - M_t^n|\big] \leq \alpha \int_0^T \mathbb{E}\big[|M_s^n - M_s^{n-1}|\big]\, ds,$$

and with the same arguments of the proof of Proposition 9 we can find a positive constant C, depending on x_0, m_0 and T, such that

$$\mathbb{E}\big[|X_t^{n+1} - X_t^n|\big] \leq C \int_0^T \mathbb{E}\big[|X_s^n - X_s^{n-1}| + |M_s^n - M_s^{n-1}|\big]\, ds.$$

We deduce that $\sum_n \sup_{[0,T]} \mathbb{E}\big[\big(|X_s^n - X_s^{n-1}| + |M_s^n - M_s^{n-1}|\big)\big] < \infty$, therefore there is a continuous adapted process $(X_t, M_t)_{t\geq 0}$ such that

$$\lim_{n\to\infty} \sup_{[0,T]} \mathbb{E}\big[\big(|X_s^n - X_s| + |M_s^n - M_s|\big)\big] = 0.$$

This convergence in L^1 implies that $(X_t, M_t)_{t\geq 0}$ is a solution to (25)-(26) and concludes the proof.

□

6 Quantification of the mean-field convergence

In this last section we use the coupling method to quantify the convergence of the empirical laws μ_N towards the law of the unique process that solves (7) and (8). We start by noticing that, thanks to Theorem 3, there exists a family of stochastic processes

$$((Y_t^{1,N}, P_t^{1,N}), \ldots, (Y_t^{N,N}, P_t^{N,N}))_{t\geq 0}, \tag{27}$$

such that

$$Y_t^{i,N} = X_0^{i,N} + t - \int_0^t Y_{s-}^{i,N} \int_0^\infty \mathbf{1}_{\{u \leq a(Y_{s-}^{i,N}, P_{s-}^{i,N})\}} \mathcal{N}^i(du, ds),$$

and

$$P_t^{i,N} = M_0^{i,N} - \alpha\Big[\int_0^t P_s^{i,N}\, ds - J\int_0^t \int_0^s \mathbb{E}[a(Y_{s-w}^{i,N}, P_{s-w}^{i,N})]\, b(dw) ds\Big],$$

where the initial conditions and the Poisson processes are exactly as described in the introduction. In the following, we use the shorthand notation $\eta_N(t)$ for the empirical mean associated to the exchangeable family $(Y_t^{i,N}, M_t^{i,N})$.

By the definition of $X_t^{i,N}$, it follows that

$$\begin{aligned} X_t^{i,N} - Y_t^{i,N} = \int_0^t \int_0^\infty \Big[- X_{s-}^{i,N} \mathbf{1}_{\{u \leq a(X_{s-}^{i,N}, M_{s-}^{i,N})\}} \\ + Y_{s-}^{i,N} \mathbf{1}_{\{u \leq a(Y_{s-}^{i,N}, P_{s-}^{i,N})\}}\Big] \mathcal{N}^i(du, ds), \end{aligned}$$

then

$$\begin{aligned} \mathbb{E}\big[|X_t^{i,N} - Y_t^{i,N}|\big] \leq \int_0^t \mathbb{E}\Big[|X_s^{i,N} - Y_s^{i,N}|\, a(X_s^{i,N}, M_s^{i,N}) \wedge a(Y_s^{i,N}, P_s^{i,N})\Big]\, ds \\ + \int_0^t \mathbb{E}\Big[(X_s^{i,N} \wedge Y_s^{i,N})\big|a(X_s^{i,N}, M_s^{i,N}) - a(Y_s^{i,N}, P_s^{i,N})\big|\Big]\, ds. \end{aligned} \tag{28}$$

Since initial distribution are compactly supported, we notice that $Y_t^{i,N} \leq A(t)$ and $M_t^{i,N} \leq B(t)$ for some locally bounded functions A, B independent of N. Moreover, for any $i \in \mathbb{N}$ we also have that $X_t^{i,N} \leq A(t)$ and

$$\mathbb{P}(\sup_{[0,T]} M_t^{i,N} \geq C_M) \xrightarrow{N\to\infty} 0,$$

exponentially fast. Therefore, conditional to the event $e(M^N) = \{\sup_{[0,T]} M_t^N \leq C_M\}$, a is bounded and Lipschitz continuous (with a Lipschitz continuity constant independent of N) in both variables. For any $i = 1, \ldots, N$, we conclude that

$$\begin{aligned} \mathbb{E}\big[|X_t^{i,N} - Y_t^{i,N}|\big] \leq C_T \int_0^t \mathbb{E}\big[\big(|X_s^{i,N} - Y_s^{i,N}| + |M_s^{i,N} - P_s^{i,N}|\big)\big]\, ds \\ + C_T \mathbb{P}\big[\mathbf{1}_{e(M^N)^c}\big]. \end{aligned}$$

for some positive constant C_T independent of N. Similarly, by the definition of $M_t^{i,N}$, we have

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[|M_t^{i,N}-P_t^{i,N}|\big] \leq \frac{\alpha}{N}\sum_{i=1}^{N}\int_0^t \mathbb{E}\big[|M_s^{i,N}-P_s^{i,N}|\big]\,ds$$
$$+\frac{\alpha J}{N}\sum_{i=1}^{N}\int_0^t \mathbb{E}\big[|a(X_s^{i,N},M_s^{i,N})-a(Y_s^{i,N},P_s^{i,N})|\big]\,ds$$
$$+\frac{\alpha J}{N}\mathbb{E}\Big[\Big|\sum_{i=1}^{N}\Big(\int_0^{t-\tau_i} a(Y_s^{i,N},P_s^{i,N})\,ds-\int_0^t\int_0^s \mathbb{E}\big[a(Y_{s-w}^{i,N},P_{s-w}^{i,N})\big]\,b(dw)\,ds\Big)\Big|\Big], \tag{29}$$

we finally see that using the arguments of *Step 2* of the proof of Proposition 12, we notice that there is another positive constant, that we also call C_T, such that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[|M_s^{i,N}-P_s^{i,N}|\big] \leq \frac{C_T}{N^{1/2}}$$
$$+\frac{C_T}{N}\sum_{i=1}^{N}\int_0^t \mathbb{E}\big[\big(|X_{s'}^{i,N}-Y_{s'}^{i,N}|+|M_{s'}^{i,N}-P_{s'}^{i,N}|\big)\big]\,ds+C_T\mathbb{P}(\mathbf{1}_{e(M^N)^c}),$$

getting the

Proof (Proof of Theorem 5 - (compactly supported case)) Gathering (28) and (29) and using the Gronwall's lemma and the fact that $\mathbb{P}(\mathbf{1}_{e(M^N)^c})$ is going exponentially fast to 0, we get that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[\big(|X_s^{i,N}-Y_s^{i,N}|+|M_s^{i,N}-P_s^{i,N}|\big)\big] \leq \frac{C_T}{N^{1/2}}.$$

To finish, we apply Fournier-Guillin [4, Theorem 1] with $d=2$, $p=1$, $q=2+\epsilon$, to find that there exists a positive constant C independent of N such that

$$\mathbb{E}\big[\mathcal{W}_1\big(\eta_N(t),\mathcal{L}(Y_t^{1,N},M_t^{1,N})\big)\big] \leq C\,\mathbb{E}\big[(Y_t^{1,N},M_t^{1,N})^{2+\epsilon}\big]^{1/2+\epsilon}\frac{\log(1+N)}{N^{1/2}},$$

but, since the initial laws $(g_0\otimes m_0)$ are compactly supported, all polynomial moments of the solution are upper bounded by a constant independent of N. Using triangular inequality we get that

$$\begin{aligned}\mathbb{E}&\big[\mathcal{W}_1\big(\mu_N(t),\mathcal{L}(Y_t^{1,N},M_t^{1,N})\big)\big]\\ &\leq \mathbb{E}\big[\mathcal{W}_1\big(\mu_N(t),\eta_N(t)\big)\big]+\mathbb{E}\big[\mathcal{W}_1\big(\eta_N(t),\mathcal{L}(Y_t^{1,N},M_t^{1,N})\big)\big]\\ &\leq C_T\frac{\log(1+N)}{N^{1/2}},\end{aligned}$$

since $\mathcal{S}$ has only one element we conclude that locally in time $\mu_N \xrightarrow{N\to\infty} \mathcal{L}((Y^{1,N}, M^{1,N}))$, as $\log(1+N)/\sqrt{N}$.

□

A General Theorems for Stochastic Processes

Remarks on the proof of Lemma 13

Let us recall that the last thing we got in the main text was that for any $\varphi \in C_b^2(\mathbb{R}_+^2)$, the process

$$\varphi(Y_t, M_t) - \varphi(Y_0, M_0) - \int_0^t \partial_m \varphi(Y_s, M_s)\Big[-\alpha M_s + \alpha\,\varepsilon \int_0^\tau \mathbb{E}[a(Y_{s-s'}, M_{s-s'})]\, b(ds')\Big] ds$$
$$- \int_0^t \partial_y \varphi(Y_s, M_s)\, ds - \int_0^t a(Y_s, M_s)\big(\varphi(0, M_s) - \varphi(Y_s, M_s)\big) ds, \quad (30)$$

is a local martingale.

Let us recall now the Jacob-Shiryaev [7, Theorem II.2.42 page 86]

Theorem 15 *There is equivalence between:*

- (Y_t, M_t) *is a semimartingale, and it admits the characteristics* $(B, 0, \nu)$*; i.e.,* (Y_t, M_t) *writes*
$$(Y_t, M_t) \;=\; (Y_0, M_0) + \mathcal{M}^c + B,$$
where $\mathcal{M}^c$ *is the continuous local martingale of the canonical decomposition,* B *is predictable and* ν *is a predictable random measure on* $\mathbb{R}_+ \times \mathbb{R}_+^2$*, namely the compensator of the random measure associated to the jumps of* X*.*
- *For each bounded function* $\varphi \in C^2(\mathbb{R}_+^2)$*, the process*
$$\varphi(Y_t, M_t) - \varphi(Y_0, M_0) - \int_0^t \partial_y \varphi(Y_{s-}, M_{s-})\, dB_s^y - \int_0^t \partial_m \varphi(Y_{s-}, M_{s-})\, dB_s^m$$
$$- \int_0^t \{\varphi(Y_{t-} + y, M_{t-} + x) - \varphi(Y_{t-}, M_{t-})$$
$$- y\, \partial_y \varphi(Y_{t-}, M_{t-}) - m\, \partial_m \varphi(Y_{t-}, M_{t-})\}\, \nu(ds, dy, dm)$$
is a local martingale.

Then, in our case of study, by choosing the characteristics

$$B_t^y \;=\; \int_0^t \Big[1 + Y_s\, a(Y_{s-}, M_{s-})\Big]\, ds,$$
$$B_t^m \;=\; \int_0^t \Big[-\alpha M_s + \alpha\,\varepsilon \int_0^s \mathbb{E}[a(Y_{s-s'}, M_{s-s'})]\, b(ds')\Big]\, ds,$$

and by

$$\nu(ds, dy, dm) \;=\; a(Y_{s-}, M_{s-})\, ds\, \delta_{-Y_{s-}}(dy)\delta_0(dm),$$

we get that (Y_t, M_t) is a semi martingale.

As for the second important Jacob-Shiryaev [7, Theorem III.2.26 page 157] cited in the main text, let us now rewrite their general result to our study case. Consider the stochastic differential equation

$$\begin{cases} (Y_0, M_0) \;=\; (\xi_y, \xi_m) \\ d(Y_t, M_t) \;=\; \beta(t, Y_t, M_t)\, dt + \delta(t, Y_{t-}, M_{t-}, z)\big(\mathcal{N}(du, dt) - q(du, dt)\big), \end{cases} \quad (31)$$

where $\mathcal{N}$ is a standard Poisson process with intensity measure $q(du, dt) = du\, dt$.

Theorem 16 *Let η be a suitable initial condition (i.e., a probability on $\mathbb{R}_+^2$), and β, δ be*

$$\begin{cases} \beta = (\beta^1, \beta^2), \text{ a Borel function: } \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+^2, \\ \delta = (\delta^1, \delta^2), \text{ a Borel function: } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+^2. \end{cases}$$

The set of all solutions to (31) with initial condition η is the set of all solutions to a martingale problem on the canonical space where the characteristics $(B, 0, \nu)$ are given by

$$B_t^i(w) = \int_0^t \beta^i(s, Y_s(w), M_s(w))\, ds, \;\; \nu(w, dt \times dy\, dm) = dt \times K_t(Y_t(w), M_t(w), dy, dm),$$

with

$$K_t(y, m, A) = \int_0^\infty \mathbf{1}_{\{A \setminus \{0\}\}}(\delta(t, u, y, m))\, du.$$

We notice that (Y_t, M_t) indeed solves the Martingale problem given by (30), therefore it is a solution to the equation (31) for some standard Poisson process, and therefore Lemma 13 is proved.

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